

Lectures on Cohomology &
Coxeter unip rep.s (at SUSTech)

L

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① Plan

AIM: Study Lusztig's 1976 paper:
"Coxeter orbits and eigenspaces of Frobenius"
for $GL_3(\mathbb{F}_q)$ (& $GL_2(\mathbb{F}_q)$).

Some references:

(rep.s) Deligne-Lusztig's 1976 paper

Bonnafe, Carter, Digne-Michel,
Geck-Malle, Srinivasan's books.

Lusztig's slim white book (1977)

(cohom) Milne's lectures/book on étale cohomology.
Freitag-Kiehl's & Tamme's books.
SGA4½.

② A brief recall (of DL)

(2)

G - connected reductive group / $\mathbb{F}_q$.

$$G := G \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

$$\left\{ \begin{array}{l} \text{absolute Frob } F_{\text{abs}} : G \rightarrow G \\ \quad (\text{on } \mathcal{O}(G) : f \mapsto f^q). \\ \text{generator of } \text{Gal}(\mathbb{F}_q) : \varphi : \overline{\mathbb{F}_q} \xrightarrow{\sim} \overline{\mathbb{F}_q}. \end{array} \right.$$

$$1) F := F_{\text{abs}} \times \text{Id}_{\overline{\mathbb{F}_q}} : G \rightarrow G.$$

(geom Frob, bijective, not autom.)

$$2) \text{Id}_G \times \varphi : G \rightarrow G.$$

(arithmetic Frob, automorphism of $\text{sch}/\mathbb{F}_q$.)

This week, by Frobenius, we mean F .

Def : $L : G \rightarrow G ; g \mapsto g^{-1} \cdot F(g)$

B - F -stable Borel of G . (3)

$$U := R_u(B)$$

T - F -stable maximal torus $\in B$.

$$W := N_G(T)/T \xrightarrow{\sim} B \backslash G/B$$

Def: $R_{w,u}^\theta = \sum_i (-1)^i \cdot H_c^i \left(L^{-1}(wU), (\bar{\mathbb{Q}}_\ell)_\theta \right)$,
 $\theta \in \text{Irr}(T^{wF})$, $w \in N_G(T)^F$. $\downarrow$
1x2.

$\downarrow$
 $wF: t \mapsto {}^{wF}t$

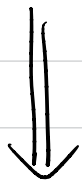
Thm (Deligne-Lusztig):

1. Every irrep of G^F appears (up to ± 1) in $R_{w,u}^\theta$ for some (θ, w, U) .
2. $R_{w,u}^\theta$ is independent of U , and for w it depends only on the image in W .
3. If θ is in general position, $R_{w,u}^\theta$ is (up to ± 1) irreducible.

(This week: $\theta=1$, $G = GL_2, GL_3$.) ⁽⁴⁾

③ Reformulation for $\theta=1$. (Carter Chp-7.7)

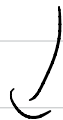
$L^{-1}(wU)$ is a vector bundle over
a T^{wF} -cover of $L^{-1}(BwB)/B$.



Spectr seq.

$$R_{w,u}^1 \cong \sum_i (-1)^i H_c^i \left(\underbrace{L^{-1}(BwB)/B}_{X_w^{!!}}, \overline{\mathbb{Q}}_\ell \right).$$

$X_w \cong$ the subvariety $\subseteq G/B$ consisting of
Borels B' s.t. $(B', FB') \in G \cdot (B, {}^wB)$.



relative position at w .

④ Type A case

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$$G = GL_n.$$

$$\left\{ \begin{array}{c} \text{Borels} \\ \text{of } G \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{complete flags} \\ \text{of } V = (\overline{\mathbb{F}_q})^n \end{array} \right\}$$

Rank: $\mathcal{F} : V_i, \mathcal{F}' : V'_i$ two complete flags.

Then "in relative position w ":

$$\dim V_i \cap V'_j = \# \left(\{1 \dots i\} \cap \{w(1) \dots w(j)\} \right) \quad \forall i$$

Example:

$$n=2, w=(12): \quad \mathcal{F} = V_1 \subseteq V_2 \in X_w$$

↓

$$\dim V_1 \cap FV_1 = 0$$

$$\Leftrightarrow X_w = \mathbb{P}^1 \setminus \mathbb{P}^1(\overline{\mathbb{F}_q}).$$

$(xy^2 - x^2y - 1 = 0)$ is a μ_{q+1} -cover of X_w .

$$n=3, w = (1\ 2\ 3):$$

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$$V = V_1 \oplus FV_2 \leftarrow \begin{bmatrix} \dim V_1 \cap FV_1 = 0 \\ \dim V_1 \cap FV_2 = 0 \\ \dim V_2 \cap FV_1 = 1 \\ \dim V_2 \cap FV_2 = 1 \end{bmatrix} \begin{matrix} \downarrow \\ V_2 = V_1 \oplus FV_1 \\ \uparrow \end{matrix}$$

$$V = V_1 \oplus FV_1 \oplus F^2V_1 \dots (*)$$



V_1 is not on any F -stable plane.

$$\Rightarrow X_w = \mathbb{P}^2 \setminus \text{F-stable lines}$$

For general $n, w = (1\ 2\ 3 \dots n):$

$$X_w = \mathbb{P}^{n-1} \setminus \text{F-stable hyperplanes}$$



Proposition. For $GL_n, w = (1 \dots n)$, X_w is an irr smooth variety of $\dim = n-1$.

For general n with $w=(1 \dots n)$, $(*)$

[7]

holds as well, so:

$$\begin{aligned}
 X_w &= \left\{ [x_1 \dots x_n] \in \mathbb{P}^{n-1} \mid \det \begin{pmatrix} x_1 & x_1^q & \dots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \dots & x_2^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^q & \dots & x_n^{q^{n-1}} \end{pmatrix} \neq 0 \right\} \\
 &= \left\{ (x_2 \dots x_n) \in \mathbb{A}^{n-1} \mid \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_2 & x_2^q & \dots & x_2^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^q & \dots & x_n^{q^{n-1}} \end{pmatrix} \neq 0 \right\} \\
 &\quad \text{(principal open)}
 \end{aligned}$$

$\Downarrow$

Prop: For GL_n , $w=(1 \dots n)$, X_w is affine.

Prop: For GL_n , $w=(1 \dots n)$, $X_w^{F^i} = \emptyset$, $i=1 \dots n-1$

One purpose: Compute $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$

for GL_3 , $w=(1 \ 2 \ 3)$, as modules
of $GL_3(\mathbb{F}_q) \times \langle F \rangle$, for each
single i .

⑤ "Elementary" ℓ -adic cohomology 18

X - a variety / $\overline{\mathbb{F}}_q$

$\downarrow$
For us: reduced quasi-proj sch / a field.

ℓ - a prime $\neq p = \text{char}(\overline{\mathbb{F}}_q)$.

Two classes of sheaves:

1. $\mathcal{F}_m := \mathbb{Z}/\ell^m$, the constant étale sheaf on X .
($m \in \mathbb{Z}_{>0}$)

$\searrow$
This means that:

$$"U \xrightarrow{\text{étale}} X" \rightsquigarrow \mathcal{F}_m(U) := (\mathbb{Z}/\ell^m)^{\pi_0(U)}$$

(restriction maps are diagonal identity map wrt conn components.)

2. $X \xhookrightarrow{j} \widehat{X}$ an open immersion with dense image,
 $\widehat{X}$ - a proj. variety.

($\widehat{X}$ exists as our X is q -proj. so no need of Nagata.)

$j_! \mathcal{F}_m$ - extension by zero sheaf on $\hat{X}$. $\hookrightarrow$

$$\Downarrow$$

$$U \xrightarrow[\text{étale}]{\varphi} \hat{X}$$



$$j_! \mathcal{F}_m(U) = \begin{cases} (\mathbb{Z}/\ell^m)^{\pi_0(U)} & , \varphi(U) \subseteq j(X) \\ 0 & , \text{otherwise} \end{cases}$$

taking sheafification.

Cech thy:

$$\mathcal{U} = \{U_i \xrightarrow{\text{étale}} X, \text{ union of images} = X\}_{i \in I}$$

$$C(\mathcal{U}, \mathcal{F}_m) := \prod_{i_0, \dots, i_r \in I^{r+1}} \mathcal{F}_m \left(U_{i_0} \times_X \dots \times_X U_{i_r} \right)$$

$$C^r(\mathcal{U}, \mathcal{F}_m) \xrightarrow{d_r} C^{r+1}(\mathcal{U}, \mathcal{F}_m) \quad \text{L0}$$

$$(s_{i_1}, \dots, s_{i_r})_{i_1, \dots, i_r \in I^{r+1}} \mapsto \left(\sum_{j=0}^{r+1} (-1)^j s_{i_1, \dots, \hat{i}_j, \dots, i_{r+1}} \Big|_{\mathcal{F}_m(u_{i_1}, \dots, u_{i_{r+1}})} \right)_{\substack{\text{omitting } j. \\ i_1, \dots, i_{r+1} \in I^{r+2}}}$$

$$H^r(X, \bar{\mathbb{Q}}_l)$$

$$\varprojlim_m \left(\varinjlim_{\mathcal{U}} \frac{\text{Ker } d_r}{\text{Im } d_{r-1}} \right) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$$

$\downarrow$
 lim wrt refinements

$$(\chi_{\text{cr}} = 0)$$

$$H_c^r(X, \bar{\mathbb{Q}}_l) := \text{"replace } (X, \mathcal{F}_m) \text{ by } (\hat{X}, j_* \mathcal{F}_m) \text{ in above}$$

(independent of j !)

(a simple exercise: try compute H^0 from above def'n.)

Rank: Why start from torsion sh, not just $\bar{\mathbb{Q}}_l$? look at FK-18. for an example!

① Tools for computing cohomology.

1. X - a variety $/ \overline{\mathbb{F}}_q$, with a Frob $F / \mathbb{F}_q$.

$\Downarrow$

$H^r(X, \overline{\mathbb{Q}}_l)$ - finite dimensional spaces $/ \overline{\mathbb{Q}}_l$
 (& $H_c^r(X, \overline{\mathbb{Q}}_l)$) $(= 0 \text{ unless } r \in [0, 2 \cdot \dim X])$

2. X' - a variety $/ \overline{\mathbb{F}}_q$.

$$X' \xrightarrow[\text{univ. homeo. mor.}]{f} X$$

$\Downarrow$

$$H^r(X, \overline{\mathbb{Q}}_l) \xrightarrow[\text{linear}]{f^*} H^r(X', \overline{\mathbb{Q}}_l)$$

In general, it makes no sense to talk about f^* in this way: The functoriality needs data on schemes.

Heuristically:

$$\begin{aligned} (s_{i_1}, \dots, s_{i_r})_{I^{r+1}} &\longmapsto f^*(s_{i_1}, \dots, s_{i_r})_{I^{r+1}} \\ \prod_{i_1, \dots, i_r \in I^{r+1}} \mathcal{F}_m(\overset{\cap}{u_{i_1} \dots u_{i_r}}) &\quad \quad \quad \overset{\text{permuting the components}}{\uparrow} \quad \prod_{i_1, \dots, i_r \in I^{r+1}} \mathcal{F}_m(\overset{\cap}{u'_{i_1} \dots u'_{i_r}}) \\ = C^r(\mathcal{U}, \mathcal{F}_m) &\quad \quad \quad = C^r(\mathcal{U}', \mathcal{F}_m) \end{aligned}$$

(Similar construction holds for H_c^r .)

3. RH over $\overline{\mathbb{F}}_q$: (Deligne Weil II) L²

α — an eigenvalue of F^* on $H_c^r(X, \overline{\mathbb{Q}}_\ell)$

$\Rightarrow \alpha \in \overline{\mathbb{Q}}$, and $\exists j \in \mathbb{Z}_{[0, r]}$ s.t.

$$|\sigma(\alpha)| = q^{j/2} \quad \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

If X is smooth & proj, then $j = r$

4. Grothendieck-Lefschetz trace formula:

$$\# X(\mathbb{F}_{q^m}) = \sum_i (-1)^i \cdot \text{Tr} \left(F^{*m}, H_c^i(X, \overline{\mathbb{Q}}_\ell) \right)$$

(see SGA-4 $\frac{1}{2}$ (Rapport 3.2))

5. Vanishing for affine:

If X is affine and of pure dim = d , then

$$H_c^r(X, \overline{\mathbb{Q}}_\ell) = 0 \quad \forall r \notin [d, 2d].$$

6. Betti numbers (comparison thm)

[3]

Y - smooth projective scheme / $\mathbb{Z}$.

Then:

$$\dim_{\bar{\mathbb{Q}}_l} H^i(Y_{\bar{\mathbb{F}}_q}, \bar{\mathbb{Q}}_l) = \text{The } i\text{th Betti } \beta_i \text{ of } Y(\mathbb{C}).$$

7. Top cohomology spaces

X - smooth conn var. of $\dim = d$ / $\bar{\mathbb{F}}_q$

$$f: X \xrightarrow{\sim} X / \bar{\mathbb{F}}_q.$$

F : a geom Frobenius on X over $\bar{\mathbb{F}}_q$.

Then: (a). $f^* = \text{Id}$ on $H_c^{2d}(X, \bar{\mathbb{Q}}_l)$

(b). $F^* \curvearrowright H_c^{2d}(X, \bar{\mathbb{Q}}_l)$ by " $\cdot q^d$ ".

Rmk: In general, X - a variety of pure $\dim = d$, then:

$$H_c^{2d}(X, \bar{\mathbb{Q}}_l) = \bar{\mathbb{Q}}_l^{\text{Irr}(X)}; \quad \text{Irr}(X) = \# \text{ irr components.}$$

Any $\varphi: X \xrightarrow{\sim} X$ acts on H_c^{2d} as a permutation permuting the irr components. (Ponnet - rat pt. - 7.5.21).

Exercise: What about $H^0(X)$?

8. Poincaré duality

X - smooth connected variety / $\overline{\mathbb{F}}_q$ ($\neq \emptyset$)
($\dim = d$.)

$\Rightarrow \exists$ a non-degenerate pairing of vec sp / $\overline{\mathbb{Q}}_\ell$

$$H_c^r(X, \overline{\mathbb{Q}}_\ell) \times H^{2d-r}(X, \overline{\mathbb{Q}}_\ell) \rightarrow \overline{\mathbb{Q}}_\ell$$

Remark: 1. Here we ignored the Tate-twist symbol

2. $(F^*a, F^*b) = q^d \cdot (a, b)$

3. If $f: X \xrightarrow{\sim} X$, $(f^*a, f^*b) = (a, b)$

(Ref: Milne bk - 11.6 or
Milne lecture 24.2 (a) cf.)

$\downarrow$
implies the two reps
in the pair are dual.

9. Long exact sequence

$Z \subseteq X$ a closed subvariety

$\downarrow$
(X - a general
variety)

$$U := X \setminus Z$$

For Lusztig, X is the DL
smooth compactification while
for us X is just $\mathbb{P}^{n-1}$, much
simpler.

$\Rightarrow$ LES:

(Milne - 120)

$0 \rightarrow \dots$

$$\dots \rightarrow H_c^r(U, \bar{\mathbb{Q}}_l) \rightarrow H_c^r(X, \bar{\mathbb{Q}}_l) \rightarrow H_c^r(Z, \bar{\mathbb{Q}}_l) \rightarrow H_c^{r+1}(U, \bar{\mathbb{Q}}_l) \rightarrow \dots$$

$$\left(\begin{array}{l} \text{comes from SES: } 0 \rightarrow j_! \mathcal{F}_m \rightarrow \mathcal{F}_m \rightarrow i_* \mathcal{F}_m \rightarrow 0 \\ \text{(here } j: U \hookrightarrow X; i: Z \hookrightarrow X \text{)} \end{array} \right)$$

Rmk: $\varphi: X \rightarrow X$ a univ homeo mor. preserving U .
 $\Rightarrow \varphi^*$ acts on the LES.

$$\left(\begin{array}{l} \text{reasoning: applying } \varphi^* \text{ to the SES} \\ (\varphi^* \text{ is exact}), \text{ then use the proper base} \\ \text{change thm.} \end{array} \right)$$

10. Katz-Messing integrality

(In our setting, this should be replaced by a homotopy property.)

X - smooth proj var. / $\bar{\mathbb{F}}_q$

$$\varphi: X \xrightarrow{\sim} X$$

Thm: $\text{Tr}(\varphi^*, H^r(X, \bar{\mathbb{Q}}_l)) \in \mathbb{Z}$, and it is independent of $l \neq p$.

(Katz-Messing - 1974: Consequences of RH - Thm 2)

② Example: $GL_2(\mathbb{F}_q)$ (We assume basic knowledge of DL characters as in Digne-Michel.) (6)

Lemma: X - conn smooth proj curve / $\overline{\mathbb{F}}_q$, genus g , then:

$$H^i(X, \overline{\mathbb{Q}}_\ell) = \begin{cases} \overline{\mathbb{Q}}_\ell & , i=0 \\ \overline{\mathbb{Q}}_\ell^{2g} & , i=1 \\ \overline{\mathbb{Q}}_\ell & , i=2 \\ 0 & , i \geq 3 \end{cases}$$

pf $\approx$ study the Kummer exact seq.:

(But only need this for P' ; so can be replaced by Betti)

$$X_w = P' \setminus P'(\mathbb{F}_q) (= U)$$

$\Downarrow$ LES

$$\begin{aligned} \dots \rightarrow H_c^r(X_w, \overline{\mathbb{Q}}_\ell) &\rightarrow H^r(P', \overline{\mathbb{Q}}_\ell) \rightarrow H^r(P'(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \\ &\rightarrow H_c^{r+1}(X_w, \overline{\mathbb{Q}}_\ell) \rightarrow \dots \end{aligned}$$

$$\implies H_c^r(X_w, \overline{\mathbb{Q}}_\ell) \cong H^r(P', \overline{\mathbb{Q}}_\ell) \quad \forall r \geq 2$$

and an exact seq.:

$$0 \rightarrow H_c^0(X_w) \rightarrow H^0(P') \rightarrow H^0(P'(\mathbb{F}_q)) \\ \rightarrow H_c^1(X_w) \rightarrow H^1(P') \rightarrow 0$$

$\Downarrow$

$$H_c^r(X_w) \cong \begin{cases} 0 & , \quad r=0 \\ \overline{\mathbb{Q}}_l^2 & , \quad r=1 \\ \overline{\mathbb{Q}}_l & , \quad r=2 \\ 0 & , \quad r \geq 3 \end{cases}$$



Question: $H_c^r(X_w)$ as representations of GL_2 ?

Lemma: For $w = (12)$:

(DL)

$$(1). \langle R_{w,u}^1, R_{w,u}^1 \rangle_{G^F} = 2$$

$$(2). \langle \text{Ind}_{B^F}^{G^F} 1, \text{Ind}_{B^F}^{G^F} 1 \rangle = 2$$

$$(3). \langle \text{Ind}_{B^F}^{G^F} 1, R_{w,u}^1 \rangle = 0$$

$$(1) \Rightarrow \langle H_c^2 - H_c^1, H_c^2 - H_c^1 \rangle = 2 \text{ for } GL_2.$$

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$$\Rightarrow \text{Two cases: } \begin{cases} H_c^2 = \rho_1 \\ H_c^1 = \rho_1 + \rho_2 + \rho_3 \end{cases} \quad \text{or} \quad \begin{cases} H_c^2 = \rho_1 \\ H_c^1 = \rho_2 \end{cases}$$

$$(\rho_i \text{ irrep, } \rho_i \neq \rho_j, \quad i \neq j)$$

$$(\rho_1 = \text{trivial rep by Tip claim})$$

Moreover:

$$(2) \Rightarrow \text{Ind}_{B^F}^{G^F} 1 = 1 + \rho$$

$$= 1 + \text{St}$$

↳ The Steinberg char
of $GL_2(\mathbb{F}_q)$.

So:

$$(3) \Rightarrow H_c^1 = \rho = \text{St}.$$



$$\underline{\text{Thm}}: \text{ For } GL_2, \quad H_c^r(X_w) \cong \begin{cases} \text{the 1-dim trivial rep.} & , \quad r=2 \\ \text{the Steinberg rep} & , \quad r=1 \end{cases}$$

Question²: How does $F \curvearrowright H_c^i(X_w, \bar{\mathbb{Q}}_l)$?

Step-1: F -action & $GL_2(\mathbb{F}_q)$ -action commutes,

$\Rightarrow H_c^2, H_c^1$ are eigenspaces of F .

Step-2: First: $\# X_w^{F^i} = q^i - 2, i=1, 2, \dots$
(as $X_w = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$).

$\Downarrow$ (trace formula)

$$\# X_w^{F^i} = \lambda_2^i \cdot 1 - \lambda_1^i \cdot q, \quad i=1, 2, \dots$$

$\downarrow$
(λ_i the eigenvalue of F^* on H_c^i .)

$\Downarrow$

$$q^i - 2 = \lambda_2^i \cdot 1 - \lambda_1^i \cdot q, \quad i=1, 2, \dots$$

$$\text{So: } \begin{cases} \lambda_2 = q \\ \lambda_1 = 1 \end{cases}$$

So for GL_2 , Lusztig's thm is proved: [10]

Thm: If h_0 is the Coxeter number, then

F^* is an s.s. auto on $\bigoplus H_c^2(X_w, \bar{\mathbb{Q}}_\ell)$.

It has h_0 distinct eigenvalues

$\lambda_1, \dots, \lambda_{h_0}$; _(multi 2) and the eigenspaces

are mutually non-isomorphic irrep.s of

G^F . Further, the λ_j -space has

$$\dim = (-1)^i \frac{|G^F|}{|T^{wF}|} \lambda_j^{-1} \cdot \prod_{\substack{j' \\ j' \neq j}} \frac{1}{\lambda_j - \lambda_{j'}}.$$

(Holds for conn red gp. $/\mathbb{F}_\ell$ with conn Dynkin.)

Tomorrow: GL_3 .

① Frobenius action

$$G = GL_3, \quad w = (1, 2, 3),$$

$$X_w = \mathbb{P}^2 \setminus F\text{-stable lines.}$$

X_w affine $\Rightarrow$ only H_c^2, H_c^3, H_c^4 can be non-zero.

Proposition: F^* acts on $H_c^r(X_w, \overline{\mathbb{Q}}_\ell)$

$$\text{by: } \begin{cases} \times 1 & , r=2 \\ \times q & , r=3 \\ \times q^2 & , r=4 \end{cases}$$

Pf: By inductively removing $\mathbb{F}_q$ -lines from $\mathbb{P}^n$, then repeatedly use LES $\square$.

↓

(details should be an OK exercise?)

② Rational points

As X_w is irr, $\dim H_c^4 = 1$.

For other dim we need # $\mathbb{F}_q$ -pt.s:

$$\left. \begin{array}{l} X_w^F = \emptyset \\ X_w^{F^2} = \emptyset \end{array} \right\} \text{lecture - I}$$

Let $d_2 = \dim H_c^2$, $d_3 = H_c^3$

$$\begin{array}{l} \text{Trace formula} \\ \text{then tells:} \end{array} \left\{ \begin{array}{l} 0 = d_2 - q \cdot d_3 + q^2 \\ 0 = d_2 - q^2 \cdot d_3 + q^4 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} d_2 = q^3 \\ d_3 = q + q^2 \end{array} \right.$$

A fun double-check:

Prop: $|X_w^{F^3}| = q^6 - q^5 - q^4 + q^3 \left(= \frac{|G^F|}{|T^{wF}|} \right) \quad (3)$

Pf: $|X_w^{F^3}| = (q^3)^2 + (q^3)' + 1$

— “ $\mathbb{F}_{q^3}$ -pt.s on
F-stable lines”

An F-stable line is def.d by a homogeneous linear eqn / $\mathbb{F}_q \rightsquigarrow$ there are $\frac{q^3-1}{q-1}$ such lines

$$\Rightarrow |X_w^{F^3}| = (q^3)^2 + (q^3)' + 1 - \frac{q^3-1}{q-1} \cdot (q^3+1) + \text{“repeatedly points”}$$

Remarks:

1. only $\mathbb{F}_q$ -pt.s are repeatedly counted, as $\mathbb{F}_q$ -lines intersect only at $\mathbb{F}_q$ -points.

2. Any $\mathbb{F}_q$ -pt $\in \mathbb{P}^2$ lies on a (proj) $\mathbb{F}_q$ -line.

3. $\# \mathbb{P}^2(\mathbb{F}_q) = \frac{q^3-1}{q-1}$.

4. Each affine line $\ell \in \mathbb{A}^3$ lies on
 $\frac{\# \mathbb{A}^2 - 1}{q - 1} = \frac{q^2 - 1}{q - 1}$ affine planes $/ \mathbb{F}_q$.

$$\Rightarrow \text{"repeated pts"} = \frac{q^3 - 1}{q - 1} \cdot \left(\frac{q^2 - 1}{q - 1} - 1 \right)$$

$$\Rightarrow |X_w^{\mathbb{F}^3}| = q^6 - q^5 - q^4 + q^3. \quad \square$$



$|X_w^{\mathbb{F}^3}|$ is compatible with trace formula:

$$q^6 - q^5 - q^4 + q^3 = d_2 - q^3 \cdot d_3 + q^6.$$

Exercise: Compute $|X_w^{\mathbb{F}^2}|$ for general G and $i = 1, 2, \dots$ by a combinatorial method.

(see DL-2.3.)

$$\textcircled{3} \quad \underline{\langle H_c^i, H_c^j \rangle_{\mathbb{F}} = 0 \quad \forall i \neq j}$$

(5)

(A key lemma, for general case)

$$Z := \mathbb{P}^2 \setminus X_w \quad (\text{take the reduced structure})$$

$$\Rightarrow \dots \rightarrow H_c^r(X_w) \rightarrow H^r(\mathbb{P}^2) \rightarrow H^r(Z) \\ \rightarrow H_c^{r+1}(X_w) \rightarrow \dots$$

$$\Rightarrow 0 \rightarrow H^1(Z) \rightarrow H_c^2(X_w) \rightarrow H^2(\mathbb{P}^2) \\ \rightarrow H^2(Z) \rightarrow H_c^3(X_w) \rightarrow 0 \\ \dots (*)$$

Lemma!: $H^2(\mathbb{P}^2) = \overline{\mathbb{Q}}_\ell$ is the trivial rep of $GL_3(\mathbb{F}_q)$.

Pf: $GL_3(\mathbb{F}_q) \curvearrowright \mathbb{P}^2$ factors through

$$\mathrm{PGL}_3 / \mathbb{F}_q \hookrightarrow \mathbb{P}^2,$$

connected alg gp

$\Rightarrow$ By the homotopy pr. property
the assertion holds. $\square$

(DL-6.4)

an irrep of
 $\dim = q^2 + q$.

Lemma²: (1) $H^2(Z) \cong \mathrm{Ind}_{\mathrm{PF}}^{G^F} 1 \cong 1 + \chi_{q^2+q}$.

P a max. F -stable parabolic

(2) $\dim H^1(Z) = q^3$.

Pf: (1): property of top cohomology

+ Bruhat decomposition

(2): (1) + counting dim along (*) $\square$

$$(*) + \text{Lemma}^{1,2} \Rightarrow \begin{cases} H_c^2(X_w) \cong H^1(Z) \\ H_c^3(X_w) \cong \chi_{q^2+q} \end{cases}$$

Lemma³: $\langle R_{w,u}^1, R_{w,u}^1 \rangle_{GL_3(\mathbb{F}_q)} = 3$. □

(DL)

$\Rightarrow$ 3 possibilities of $H_c^2(X_w)$:

1. $St \otimes \chi_1$

2. $\chi + \chi' + \chi_{q^2+q}$ (χ, χ' are irrep.s of a suitable dim)

3. $\chi + 2\chi_{q^2+q}$

Possible Irrep dim of $GL_3(\mathbb{F}_q)$: (Steinberg, 1951 paper)

1, q^2+q , q^2+q+1 , q^3-q^2-q+1 , q^3-1 ,
 q^3 , $q(q^2+q+1)$, $(q+1)(q^2+q+1)$

$\Rightarrow \begin{cases} H_c^2(X_w) = St \otimes \chi_1, & q > 2 \\ H_c^2(X_w) = St \otimes \chi_1 \text{ or } \chi'_1 + \chi''_1, & q = 2. \end{cases}$

$\Rightarrow H_c^2(X_w) = St \otimes \chi_1, \quad \forall q$. (any 1-dim irrep factors through det)

exercise: Show that $H_c^2(X_w) \cong H^1(Z) \cong St$.

(hint: study the principal series & use inner prod.)

Rmk: For general GLn, w Coxeter: (8)

$$H_c^{\ell(w)}(X_w) \cong St, \quad H^{\ell(w)-1}(\mathbb{Z}) \cong St + \frac{1-\ell(w)}{2} \cdot 1.$$

(see Deligne-Orlik-Rapoport: Period domains... (3.3.9))

(Stop here?)

④ Groups ($\mathbb{Z}$) over ($\mathbb{Z}$) local ($\mathbb{Z}_p$) rings ($\mathbb{Z}_p^k$)

$\mathbb{Q}$ - char, uniformiser π .

$\mathbb{F}_q$ - res. fi.

G - connected red gp / $\text{Spec}(\mathbb{Q})$.

AIM: Study $\text{SmoothRep}(G(\mathbb{Q}))$

$\updownarrow$ 1:1

" $\varprojlim$ "
 $\varprojlim_{r \in \mathbb{Z}_{>0}} \text{Rep}(G(\mathbb{Q}/\pi^r))$

$$B_r = T_r \times U_r - \text{Bml of } G_{\mathcal{O}^w/\mathcal{U}^r} \quad [9]$$

1. Greenberg: G_r, B_r, U_r, T_r can be viewed as alg gp.s over $\overline{\mathbb{F}_q}$.

2. $\exists$ geom Frob F on G_r s.t.

$$G_r^F \cong G(\mathcal{O}/\mathcal{U}^r).$$



Lang map: $L: g \mapsto g^{-1}F(g)$ on G_r .

(assume: $FT_r = T_r$)

Definition: $R_{T_r, U_r}^{\mathcal{O}} := \sum_i (-1)^i H_c^i \left(L^{-1}(F_{\mathcal{U}}) \right)_0$
a virtual rep of $G_r^F \cong G(\mathcal{O}/\mathcal{U}^r)$.

Theorem: (Lusztig for char > 0 , Stasinski in general).

If $\mathcal{O}$ is regular enough, then $\pm R_{T_r, U_r}^{\mathcal{O}}$ is irreducible.

Pr.blem: Further quantitative & qualitative properties are not well-understood.
(e.g. ± 1 ?, $\dim$? ...)

Thm (C. - Stasinski, 2023?)

$r > 1$ & $\mathcal{O}$ "generic enough":

$$R_{T_r, \mathcal{O}_r}^{\mathcal{O}} \cong \pm 1 \cdot \text{Ind}_{T_r^F}^{G_r^F} P_0$$

"larger reduct-ker"

where P_0 is an irrep uniquely determined by:

$$1. \quad P_0 \mid \text{Ker}(T_r^F \rightarrow T_1^F) \text{ "smaller reduct-ker"} \\ = \underbrace{d}_{\sim} \cdot \widetilde{\mathcal{O}} \mid _$$

$$2. \quad P_0 \mid_{T_r^F} = \pm 1 \cdot \underbrace{St^{0,2}}_{\substack{r \text{ even: } 0 \\ \text{odd: } 1}} \mid_{T_r^F} \otimes \mathcal{O}.$$

Remark: 1. even r : much easier: 2. 1T!

2. " ± 1 " are determined for $q > 5$.
(q in $\mathbb{F}_q$)